**Weiner Process**

The Wiener process forms the basis for my purposes in Stochastic Calculus. So I guess we’ll start here.

**Scalar Weiner Process**

Let’s specialize to the case where the probability distribution, rather than just the moments, of the driving random variable is known. We’ll presume it’s white noise, so that:



The old formulas follow:



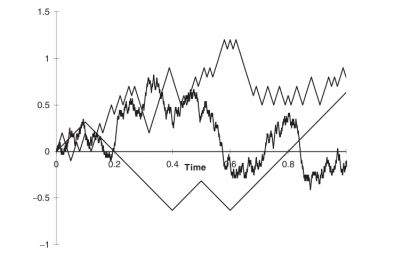
We can write this another way, defining w = dW/dt,



From this we can define the Wiener process whose distribution would follow from the definition.



Because the Wiener process can be expressed as the sum of n = anything, independent and identically distributed variables, it is considered and **infinitely divisible process**. It looks something like this:



And relatedly ΔW = W(a+Δt) – W(a), has distribution



And correlations between different segments would be the following, where Δtoverlap = the overlap of the time intervals Δt and Δt’.



**Algebraic Functions of Weiner Process**

Now let’s consider functions of W(t), i.e. F(W(t)). This is of course similar to the thing in Mello’s notes. This is also a random variable, and we’d like to ascertain its distribution. First we’ll consider algebraic operations on W(t). If we have F(t) = bW(t) and we want to determine average, variance, and p.d.f. of F(t). Well,



How about a linear relation:

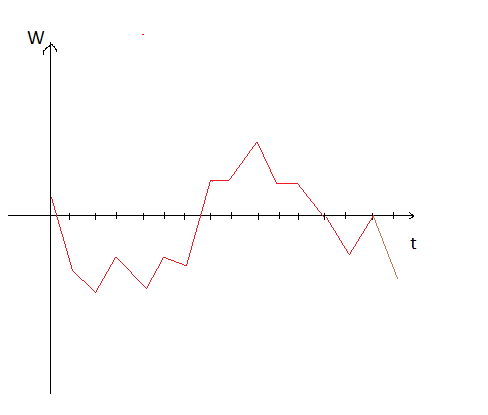


and what if we had a simple quadratic, F(t) = bW(t)2? Well in that case let’s remember that W(t) is also a Gaussian variable with average 0 and st.dev. Dt, and so we can compute statistics of this variable using the ‘GF’ techniques in the Appendix.



**Integral functions of Weiner process**

OK, and now for integration. Consider a Weiner process W(t) illustrated below. The process looks discretized, but we should have in mind a continuous one.



If we were to naively define an integral from 0 to t, we would chop the interval up into times tj, and write:



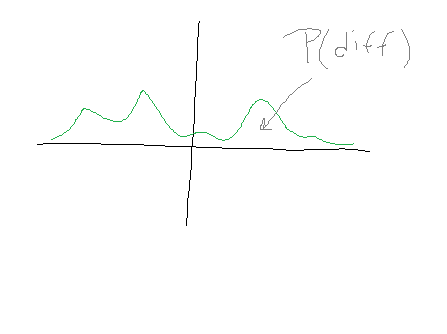
where tj\* is some time inbetween tj and tj+1. We will propose some formula for this in terms the end points, like this:



Where F is the anti-derivative of f w/r to W, and Remainder, as we’ll see, is not zero, but depends on where t\*j+1 is, inside the interval [tj, tj+1]. This is at odds with a regular Reimann sum, whose value does not depend on this detail. Equality is adjudicated in the sense that the integral and the anti-derivative have the same probability distribution (‘cause note that each are just random variables). The easier way to enforce ‘same probability distribution’ is to require the expectation of their square difference to be zero, so that F is considered equal to ∫fdW if:



(in the limit that the partition size goes to zero). To get a sense of this criterion, consider two independent variables x and y. They are mean square equal if: <x2> + <y2> - 2<x><y> = 0. If these had same mean and variance, then we’d have 2<x2> - 2<x>2 = 0, which would imply that their variance were 0, which would be a delta distribution – a pretty stringent condition. It basically means they can’t be different over any finite range because if they were, then that difference would contribute to the expectation and give a non-zero result. For instance, as can see, the only way to get < > = 0 is for P(difference) = 0 everywhere, except at perhaps a finite/denumerable # of *points*, strictly speaking.



Now note that the normal rules of calculus don’t apply here. For instance consider:



This cannot be true, as evinced by the non-equivalence of their expectations.



If we make tj\* = tj, then because the increment dW = W(tj+1) – W(tj) is independent of W(tj) itself, the expectation would be zero. Or another way to look at it is:



So from this perspective, we get zero not because the ΔW is independent of W, but because both terms give the same (up to a differentially small amount) the same thing. Anyway, if we were to place tj\* in between tj and tj+1, then they wouldn’t be independent, and so we probably wouldn’t get 0. Let’s consider the value of this integral in more detail. Let tj\* = tj + λ(tj+1 – tj), where λ ∈ [0,1]. Then…(basically we will want to break this expression up into all of its independent increments)…



Here we observe that the last term is the same as the integral itself. And so we have:



The last two terms are independent Gaussian deviants which we can write, in terms of normalized Gaussian variables ε1j and ε2j. It seems to me that this would be an exact replacement too (?), since the deviants are the sum of Gaussian variables, dW, which is therefore itself Gaussian, and will have the posited variance.



Now is it ok to replace these random terms by their expectations? If so then we’d get:



Let’s check if this is ok in the mean square sense.



Now we need only consider expectation from cross terms in that term there, as the diagonal ones will involve (ti+1 – ti)2 = (Δt/N)2, which when summed from i = 0 to N will still leave us with a factor of Δt2/N, which will go to zero in the large N limit. So then,



In the second line we note that we aren’t including i = j terms and so are missing 1 of the N terms in the sum. But this is negligible since it’s of order 1/N. So this checks out. So we can see that the result of our integration does depend on our choice of tj\*. λ = 0 corresponds to Ito’s version, and λ = ½ corresponds to Stratonvich’s version. Note that Stratonovich version obeys the ordinary rules of Calculus. Interesting that the result is completely deterministic. Let’s consider the integration more closely. First of all,



And now consider the antiderivative F(W,t) for which ∂F/∂W = f. Then let’s consider a Taylor series of F. I think we *could* series it out directly from t = 0 to t = t? Certainly in the discrete case, the resulting function would still be well defined. But in the present m.o. we’re going to chop up the time interval into pieces. And I imagine that as we shrink the time interval down, all higher order terms will vanish naturally.



Now it seems that we usually replace the ΔW2’s with the equivalent DΔt. But we’ll observe that while one could more or less do this for the first term in the brackets, since that ΔW2 is independent of the F´´ term (because the F´´ term has dW’s up to time k+λ whereas the ΔW2 term, after subtraction is taken account of, has dW’s between times k+λ and k+1), we cannot do this with the second ΔW2 term, since there will be overlap between its dW’s and the F´´’s dW’s. Still I think it’s justified to make the DΔt replacement because if we expand F´´ about tk, instead of tk+λ, only the first term in the sum ought to contribute. Well let’s check. Doing the mean square thing between the discrete integral and the proposed replacement requires the following condition:



It seems that the only term that has the possibility to survive the limit Δtk ~ Δt/N → 0 is indeed the one with just the zeroth order g term. In this case it looks like the equality will survive. If so, then we’d have:



As usual we can stop at this order in the expansion, and so then we have our result: I guess you could say it’s the **Fundamental Theorem of Stochastic Calculus**.



Note that we do not have an independent way of evaluating any of the integrals on the RHS if they have any dependence on W. The only thing we ‘know’ is the LHS, which is simply the antiderivative of f (w/r to W). Solving for the integral term we have, in other notation:



The last two integrals are ‘normal’ Riemann integrals. I wonder if there is reason to do line, surface, or volume integrals in this context.

**Example 1**

Let’s do the one we started out with:



**Example 2**

Or how about just a function of t?



**Example 3**

More difficult ones are harder to compute. Looks like we can’t do powers of W in general…



But their expectations ought to be calculable,



Note there *are* formulas for the values of these integrals. We can introduce the Ito (λ = 0), and Stratonovich (λ = ½) integrals. These would be:



And note these are related via:



We can connect them directly, via the following heuristic manipulations (see Appendix):



**Differential functions of Wiener process**

Now let’s go backwards, sort of. Consider a function F(W(t)). And let’s consider what its derivative might mean. We’ll define the derivative as:



Tnen we’ll expand the numerator about some point within the interval:



Making those replacements of W → D is really only justified within the context of integration. But…and then dividing both sides by tk+1 – tk, we have:



So then our formula is:



So the λ would govern where, in between the (t,t+dt) interval, the partial derivatives are evaluated. The Ito version evaluates at the left end point, for which λ = 0, and the Stratonovich version evaluates at the midpoint, for which λ = ½, and the last term vanishes. Product, and chain rules follow – see the vector case:



**Vector Wiener Process**

Let’s generalize to a vector Weiner process, d**W**(t). It’s pdf should be something like:



The old formulas follow:



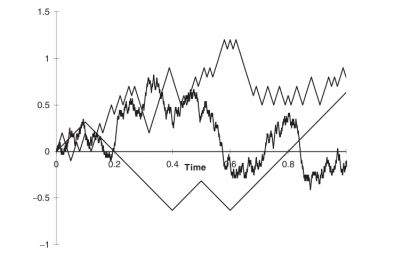
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And correlations between different segments would be the following, where Δtoverlap = the overlap of the time intervals Δt and Δt’.



**Integral Functions of Wiener process**

Taking a look below, at the ‘differential functions of W’ section, we have:



Integrating w/r to t, I think we may conclude that:



**Example**

Let’s consider the function F(W1,W2) = W1W2. Then,



So I guess you could say:



But is there any way to solve for ∫W1dW2 by itself? For instance, can we say something like:



?? Let’s consider the expectation of these integral, in the Stratonovich form. So we have:



Can this be proven? So, uh,



Gonna separate out parts that correlate with ΔW2:



Well, average is zero still souer that’s good.



The first term will go to zero because only j = i will have a non-zero net value. But then the (dt)2 will bring it down to zero. So then we’re left with the other three terms. Basically we need cross terms to be non-zero for the equality not to hold. So let’s just look for non i = j terms.



Let’s look at the first one:



This is just going to go to zero.



Well this also should go to zero. And the last:



note tbelow,i includes ti-1/2, and tabove,i includes ti+1. For the first set of four, if i < j, then ε2(tj) will be unmatched. Same if i > j. What about last term? if i > j, then everything will vanish because there will be a lowest unmatched ε2(tj) or ε2(tj+1/2). Same with i < j. What about the middles though? Take the 3rd. Seems that if i < j, there is likelihood of this term surviving. So let’s work it out:



So the two brackets on the left must match one of the four products in the rightmost bracket. Let’s set i = 10, and j = 20. Then:



And the surviving terms are:



So this seems to viable (and bad). And the whole thing seems to be symmetric w/r to interchange of i and j, so the other guy shouldn’t cancel this one out. So it appears this identity does not hold.

**Differential Functions of Weiner process**

I’m just going to consider derivatives for now. We’ll define:



and then, expanding the numerator about some point within the interval…



Equating the first and last lines, dividing by tk+1 – tk we appear to have:



We can see that the value of the derivative depends on λ of course. We can write this more succinctly as



So the λ would govern where, in between the (t,t+dt) interval, the partial derivatives are evaluated. The Ito version evaluates at the left end point, for which λ = 0, and the Stratonovich version evaluates at the midpoint, for which λ = ½, and the last term vanishes. What would a product rule look like? This also follows quite simply from the definition of dF/dt itself, since:



So we have:



Another way is to attempt this is:



which doesn’t match, interestingly, accept in the Ito case, and for a symmetric Dij. So doing the differential approach is not especially generally true. And it fails utterly for the Stratonovich derivative, which itself preserves the normal rules of calculus. What about the chain rule? Consider F(W,t) = F(G(W,t)). And again, the fundamental rule will suffice:



We could say:



Again, the Stratonovich version preserves normal calculus of course. But the Ito version adds an extra term.

**Appendix 1**

Next question is…is this a legitimate replacement?



We have to check,



Now what about the i ≠ j terms?



This requires:



So it works, and with the expected value of c. So, looking ahead, it seems we can write:



\* only the overlap between the two random variables matters, and this is why we can change Wi+1 – Wi to Wi+λ – Wi.

**Appendix 2**

Now let’s consider something else. I’ll say 0 < λ < λ´ < 1. Is the following true?



Well,



Continuing,



So we must check,



If the off diagonal terms go to zero, then the whole thing will as well. So we just need,



Interesting. So,



Apparently, only the overlapping ΔW’s matter, and so the part from λ to λ´ is irrelevant. So I’d reckon that we can say:



This probably should be enshrined.

**Appendix 3**

Now let’s consider something else again, where 0 < λ < λ´ < 1. What can we do with this?



Well,



And so we see how our highlighted result makes this manipulation into an average-able form, quite easy.