**Reducing FP equation from higher to lower manifold**

So say we have a FP equation for a set of variables Y which roam some manifold. And then say we want to consider a subset of this manifold, parameterized by variables Y(X). Does it make sense to even have a FP equation in terms of more d.o.f. than there actually are? I think the answer to this is yes. I believe the result, P(Y), however, will have a functional form that only depends on the constrained variables, like a radial probability distribution function might be simply P(x,y) = exp(-x2 – y2) = exp(-r2). Well if so, can we write down a FP equation for X, given the one we know a propos Y? It seems like it should be possible. Consider general N2L in terms of Cartesian coordinates. But if we set x = rcosθ and y = rsinθ for instance, into the equations, then we do get legit equations on a circle. So shouldn’t we be able to write down a general P(x,y) FP equation for some x-y process, then write the process in terms of θ, constrained to a circle, or whatever, and then get a commensurate P(θ), which could be deduced directly from P(x,y) using the δ function solution?

**Important Relationship between derivatives and δ function**

Remember, if f(x,y) = f(x-y), then,



where ʹ means derivative with respect to the argument. And further, if y(φ), then:



How handle derivatives?



More concise?



Really, I should be defining p(x,y) as ∫p(φ)δ(x-φ)δ(y-φ), or something. So let’s try it again,



Perfect. What about second derivatives?



So perfect again. So it would seem we can employ constraints after the fact, or employ it before the fact.

**Attempt to work it out in general**

Say we had some vector variables Yi with recursion relation: Yi+1 = Yi + δY,i(Yi,ΔWi). But that it was really a function of some lower dimensional variable Xj, such that Yi = gi(Xj), which follow their own recursion relations: Xi+1 = Xi + δX,i(Xi,ΔWi). We could write down an probability distribution equation for P(Y) and for P(X). I guess question is, would the equation for Y reduce to the equation for X, under the substitution? Equation for Y is (assuming only 2nd order is necessary)



and equation for X is:



Now does the top equation reduce to the bottom equation, upon substitution? What’s the relationship between the two P’s?



and on the RHS?



**Example 1**

Consider following (Ito) example,



We would suppose that:



and so,



On other hand, if break into real/imaginary parts, then:



We would suppose that:



and so,



But then let’s suppose that Z = eφ → φ = ln(Z). Then,



And so its PDE would be:



**Are P(z,z\*) and P(x,y) FP equations equivalent?**

Now are the complex and real PDE’s equivalent.



Converting,



**Can I convert from P(z,z\*) to P(φ) directly via δ function?**

Let’s see,



But this doesn’t seem to be reconciled. What about as a weak solution?



So a weak solution works.

**Example 2**

Let’s try this:



and so,



On other hand, if break into real/imaginary parts, then:



We would suppose that:



and so,



Now are these equivalent?



Converting,



Yes, again. Now say I define Z = Z0eφ. What would this process look like?



and so the FP would be:



Let’s set a = b2D/2 for simplicity. Then:



**What would P(x,y) equation look like from the integral approach?**

So then we’d say:



and, I just rederived the usual equation,



Woops. OK rather:



So δx and δy are:



and we have:



and then,



and,



then using a = b2D/2



and,



So finally,



compared to:



Finally, this matches.

**Can I go from P(x,y) to P(φ) via δ function?**

So, going for weak solution, and using a = b2D/2,



and,



So that checks out.

**General comment**

Seems to me that if you have a real ‘time-dvelopment operator’ acting on a complex random variable, then every term ∂/∂z will be accompanied by a ∂/∂z\* and every ∂1/∂z2 by a mirror ∂2/∂z\*2, as well as a cross term 2∂2/∂z∂z\*.  And breaking it down into real/imaginary parts would give us the same structure with x replacing z, and y replacing z\*. So I think Mello’s F(M) is really missing over half its terms.

**Example 3a**

How about,



Now in our model below, we had:



and so this would correspond to b = 1, and a = -D/2. So moments are:



and then the PDE



Let’s pull all the z’s through the derivatives, and specialize to b = 1, a = -D/2. Then,



With the value assignments, this is the same as the previous PDE. Breaking it into real/imaginary:



and so,



Now let’s use a = -D/2, and b = 1.



this also matches the previous file’s result. Now let’s say Z = eiφ, so φ = -iln(Z). And dφ = -(i/Z)dZ + (i/2Z2)dZdZ. So,



using the previous assignments, we would get dφ = dW. And this would give the desired PDE as well. What about this. Can I write a PDE in terms of X alone? Say Y = √(1-X2). Then,



In that case, the PDE would be:



And, I guess this really only holds when a = -b2D/2. So,



**Can I go from P(x) to P(φ) via δ function?**

Well, x = cosφ. And we



and then we get:



and a quick integration by parts gives us our desired result.

**Can I go from P(z,z\*) to P(φ) via δ function?**

Let’s check that the δ function substitution will work out to this:



Now let’s plug this into our PDE:



and



blah,



and,



and,



So this works. Let’s try it the other way, as a weak solution?.



next,



and,



and,



If we set a = -D/2, and b = 1, then we have:



So this also seems to work.

**What about going from P(z,z\*) to P(x,y)? Are PDE’s equivalent?**

First let’s check that the PDE’s are the same,



Converting,



So they are the same.

**Can I go from P(x,y) to P(φ) via δ function?**

Let’s see,



and,



and then,



That works out.

**Example 3b**

Suppose we have a vector which undergoes rotations. And so at any time, we may say:



where M is a rotation matrix,



And let’s say that θ is randomly distributed between -π and π. And we want to know the probability distribution of v – or rather the probability distribution of M. But could say:



And what if we considered products of matrices. Perhaps:



Then for a, b matrix:



And so,



And moments are:



So PDE for P(a,b) is:



And doing same for θ,



Then for θ matrix:



And so we need to work out how the θ has changed.



This tells us that δθ = ε, of course. And moments are:



So PDE for P(θ) is:



**Testing that one P(a,b) implies P(θ)**

Now relationship between P(θ) and P(a,b) is?



or, better?



Then does P(θ) satisfy that equation? Let’s take the temporal derivative:



and,



and,



and,



and then,



and,



So,



So it works, evidently. Does it work all the time? Can there be a general proof? Suppose we had used a symmetry in the P(a,b) equation beforehand. Would the bastardized equation still imply the P(θ) equation?



Well this has a P term, which would seem to kill it.

**Testing that P(θ) implies P(a,b)**

Now relationship between P(a,b) and P(θ) is?



Then does P(θ) satisfy that equation? Let’s take the temporal derivative:



and,



and,



and,



and,



and so that establishes what ∂P(θ)/∂t is.

**Examining if the P(a,b) equation preserves the determinant.**

So consider:



Want to determine the evolution of the expectation of the determinant,



and now,



So that’s preserved. What about all moments?



and,



and,



So all moments are zero as well, which means it’s identically zero.

**Trying to solve P(a,b) PDE and see if solution reduces to P(θ) solution**

So we have:



And doing same for θ,



It’s be nice if we could choose a new set of variables that would diagonalize the spatial operator. I guess the natural choice would be r2 = a2 + b2 and θ = tan-1(b/a). Let’s try this:



So,



and,



and,



and,



These second derivatives can be simplified a bit to:



and,



and,



Now let’s put it all together,



OK, can I simplify this wonderful equation? Let’s consider each individual derivative,



and,



Good so far. Now let’s look at the cross derivative:



Cool. Now the Pθθ guy:



as desired. And now what is this last term. Hmmmm…..



Huh. So the solution will be entirely in terms of θ = tan-1(b/a). So solution contains both a and b. So it looks like when you have a PDE in terms of variables xj, which are governed by a smaller set of quantities θk, that the solution will be in terms of P(θk(xj),t).

**Summary of results**

So we can start off with a matrix **M**, initially completely general. One could suppose that if both the specified initial condition **M**(0) and matrix model **M**ʹ that serves as the driving evolutionary ‘force’ are consistent with symmetry conditions, then **M**(z) will be as well. Furthermore, the probability distribution P(**M**) will depend on **M** solely through the reduced set of variables θk(**M**) = gk(**M**) which automatically satisfy the constraints. Additionally, the probability distribution P(**M**) will depend on the analogous probability distribution P(**θ**) via:



Therefore knowing the evolution equation of P(**M**) will enable, in principle, the determination of the evolution equation of P(**θ**). And in general if M1 = M2 = M, then ∂/∂M1 + ∂/∂M2 → ∂/∂M, and (1/2)∂2/∂M12 + (1/2)∂2/∂M22 + ∂2/∂M1∂M2 → (1/2)∂2/∂M2, etc. So basically, constraints can be imposed at the beginning of the process, or at the end. Furthermore, it seems that as long as the initial conditions and microscopic model preserve the true symmetries inherent in the problem, then the evolution on the higher manifold will as well.