**Integral Equation Approach Investigations**

**Examining how to precisely define Pδt(δX)**

Consider the process:



What are some averages:



But what is P1(X1) and P(ΔX1), P(X2)? These are?



Now question is, ‘is ∫P2(X2) = ∫∫P1(X1)PΔ(ΔX1)?’. So let’s check:



So that checks out. So the probability distribution of the increment is:



where we recognize that it may be an explicit function of all random variables that show up in the δ**X** expression, besides the Δ**W** variable. And secondly, it follows that:



Let’s reinvestigate a bit. So



Now define



The latter will have implicit X(tm), tm, Δtm dependence. So is <F(tm+1)> = ∫dXP(X,tm+1)F(X)? Note that Xʹ(tm) is a free variable, whereas X(tm) is the one which implicitly contains all the W’s. So check,



So that works out. What about <F(tm+1)> = ∫dXP(X,tm)d(ΔWm)P0(ΔWm)F(X(tm) + ΔX(tm))?



So that works. And how about <F(tm+1)> = ∫dXP(X,tm)d(ΔX)PΔ(ΔX)F(X(tm) + ΔX(tm))?



So that would check out if we can consider Xʹ and ΔXʹ to be independent. Can we? Yes.

**Checking out equivalence of two formulations of probability recursion equation**

So consider:



To perform the integration over **X**(tm) we have to introduce a Jacobian factor, because X(tm) will appear in ΔX(tm) too. So we get:



But then, if we write the whole another way, we don’t explicitly run into a Jacobian factor,



Now **X**(tm) is a part of Δ**X**(tm) here. So there is no prima facia contradiction. But are they really equivalent? I guess I’ll plug in what PΔ is, and see if it reduces to the previous equation,



Then let **X**(t) = **ξ** – Δ**X**(t) → d(Δ**X**(t)) = -d**X**(t),



and, aside from weird minus sign (maybe has to do with limits of integration) we get the same thing. So it appears the Jacobian is just obscured. This seems to be a good way to avoid Jacobians though. Here’s another proof that they’re the same:



Yeah, so why do we even bother with the Kolmorgorov equation? Analyzing some more again. But note that you have to put the PΔ**X** term within the d**X** integral, because PΔ**X** will have **X** dependence. So,



Now let’s fill in F = δ(ξ - ).



Now PΔX will depend on **X**(tm) itself.



Do Taylor series expansion:



and then we can pull out the derivatives, to get:



Let’s see if we can get the same thing from the other formulation:



But dX(t) has ξ in it, so we cannot pull derivatives past the dX(t).

**Probability distribution of slice, from SDE**

So say we have an SDE like:



what is its probability distribution?



So we’ll note the thing is Gaussian distributed. What about a vector example?



what is its probability distribution?



Seems like the ξ’s ought to be coupled to each other. But whatever for now. So again we get a sort of Gaussian distribution.

**How the typical probability distribution evolution equation**

An alternative approach to establishing the statistics of a stochastic variable is through the Kolmorgorov equation. Consider again the recursion relation:



Then for a function F(**X**) we would have:



Now introduce the probability distribution for the increment δi,



and note that Pδt(**ξ**) will depend on **X**(t) itself, if **X** is an explicit part of δ**X**. So then we can write for any function F(**X**(t)):



And then we can say:



If we specialize to: F(**X**(t)) = δ(**ξ** - **X**(t)), then we get the usual equation:



**General considerations on how one gets the Kolmorgorov equation**

There is another, ubiquitous way to write this recursion relation, that takes explicit advantage of the multiplicative recursion relation of the transfer matrix. Consider: **X**(tm+1) = **X**(tm)·**x**(tm). Writing this in terms of the probability density function p(**ξ**) = P(**ξ**)/μ(**ξ**), where dμ(**ξ**) = μ(**ξ**)dξ, we have in the continuum limit:

Typical transfer matrix example. This would be:



or if did it other way:



?? So what’s weird is that this means P follows that same formula that p is supposed to follow. But this is not true. So that means that maybe it only does this in these (Cartesian) variables? Let’s check the transformation Jacobian in polar coordinates. Will this also be 1? Might be prohibitively difficult in general, or even in N = 2 case. Is it ever the case that they’re different? Well they’re the same in the polar example.

1. initial recursive equation

2. transformation via jacobian

3. employment of delta function

4. final P equation

5. this is the same as the standard p equation.

6. they follow the same PDE regardles of whether the measure is μ(X) = 1.

But. these can’t have the same expansion regardless of whether μ(X) = 1.

Can’t see any error in these steps. So could try example and maybe i’ll see an error somewhere. Last example seems to suggest that J isn’t always 1, especially when there are fewer d.o.f. than matrix elements? So that’s where the analysis fails. More generally, I see that the actual d.o.f. in the matrix of variables will not usually be related to each other in a matrix-type fashion, as they are embedded within the matrix. But the matrix structure is quite handy to deal with. And this is why you move from a probability distribution of the actual d.o.f., X, to a probability distribution of the matrix, M, itself. How are the two related? I would guess p(M)dμ(M) = P(X)dX. Yeah I think so. A certain space of variables dX corresponds to a space of matrix dμ(M). And the probability of X being in the space is P(X)dX, which should be the same as the probability of the matrices being their space, which is p(M)dμ(M).



Yay! Except. Such a δ contraint would actually over-determine things.

**Want to start with an example process.**

Let’s use a stretch/rotate matrix as our basis:



This is a group under multiplication:



What is measure? Would be tempted to say that |M2| = Σab|Mab|2. And then,



Therefore the measure is:



which is what we’d expect. And in the other Cartesian representation, we’d just have:



So to summarize:



Now want to consider transformation Jacobian. So in Cartesian,



And in polar,



So we have:



So they’re basically the same, which suggests the Jacobian doesn’t vary w/r to ‘internal’ coordinate transformations.

Now we’d like to determine a PDE for P(a,b), or equivalently, P(λ,φ). We could model the statistics of a multiplicand like this,



Then for P(λ,φ), we’d have:



So,



And in the Cartesian representation, we’d have:



and so the equation is:



So altogether:



Should do a change of variables to make sure they are equivalent. Preliminary result:



so,



So,



and then,



and,



And finally, P(λ,φ)dλdφ = P(a,b)dadb, which implies P(λ,φ) = P(a,b)λ = P(a,b)√(a2+b2). OK, filing this in,



Well I guess i have to keep going. So,



and,



and now clear the denominator,



Combining terms,





and,



and,



and now,



It checks out – after a million mistakes.

**Point 1-4. Is the initial recursive equation correct? And are all manipulations thereafter kosher?**

So next thing is to use the Kolmorgorov equation to get the same thing. Let’s do the polar version:



Expanding this out will give the usual result. What about.



What exactly is Pδt(δλ,δφ)? Well,



So,



Now ε1 and ε2 are independent identically distributed Gaussian random variables. So we should have,



Now let’s look at it a different way. Like with the transfer matrix, these matrices follow a multiplicative recursion relation. And we can write Mʹ = Mπ·M,



What is Pδt(Mπ). Well this is – see the matrix recursion relation again,



Moving on,



Or another way:



How does this relate to the previous equation? Consider:



Now we’ll recognize that (λπ-1) and φπ are both ~ √δt. So this will truncate the expansion at:



Carrying out the averages:



And so we have:



And this matches. So recursive equation seems legit.

**Point 5. What would be recursive equation for the probability density? And is it equivalent?**

If I had done this in terms of the probability density, then:



So we get:



or,



Perturbative expansion would be:



and,



and,



and,



Let’s check this out:



And so,



So this also works out. Could’ve done this as:



**Point 6. Can the two functions, p and P, follow the same equations somehow?**

What does it take for:



to both be true:



So would need stuff to happen.

**Checking general chain of reasoning: if J were 1, would p, P equations be identical?**

Now I want to see how we would get the usual p – recursion relation, if the measure were invariant.



So invariant measure must amount to the following equality:



In general change of variables:



And in terms of P,



invariance here must require:



What is Jacobian of transformation?

**Want to start with an example process (2)**

Let’s do a single channel phase-less transfer matrix example.



We have from before.



So the Jacobian would be:



Evidently, the measure would be:



Let’s assume a multiplicand with the following statistics:



Then moments are:



And so the evolution equation would be:



so,



Or another way,



This is ugly. But let’s solve for λ in terms of λʹ, λπ. Note we need only go out to second order in λπ. We already have:



Solving for those parts:



So,



and the denominator. Oh crap.



And its inverse:



And so we’d have:



Not really looking forward to completing this calculation though. Let’s skip to the next thing: the Kol equation.



Let’s just use one of the δ functions:



Interesting that the last two are identical. Turns out we’ll need to use the last one(s), obviously, because i wouldn’t have done that unless the other ones didn’t work.



And so indeed we have:



Expanding,



Does this match?



Doing some mathcad manipulations, we see this does indeed reduce to,



How do we know we’ll always get this δ-measure result? I guess we can argue something like:





which we know is true. But then also, if we have a Haar measure, then:



**How the Kolmorgorov Equation works out?**

In the transfer matrix, context, the independent random variables **X**(t) appear within a matrix **M**(**X**(t))). And **X**(t)’s recursion relation takes a particularly clean form when written as a matrix: **M**(**X**(t+dt)) = **M**(**X**(t))·**m**(d**W**(t)). This recursive structure can be handled particularly well if we promote our collection of random variables to a random matrix. One writes P(**X**(t))d**X**(t) = p(**M**(**X**(t)))dμ(**M**(**X**(t))), and dμ(**M**(**X**(t))) = μ(**M**(**X**(t)))d**X**(t) is the ‘volume’ in matrix space corresponding to the ‘volume’ d**X**(t) in coordinate space. Introduce the following probability distribution of the multiplicand variables **x**(t), which appear in the matrix **m**(**x**(t)):



Then we can write:



or better,



We could transform this into a differential equation by writing Π-1(ξ/x) = ξ – Δ(ξ,x), and performing a Taylor series expansion of p(ξ/x,t)…



whereupon we have:



where



Typically, only the first two moments survive the limit, and so we obtain a second order differential equation.