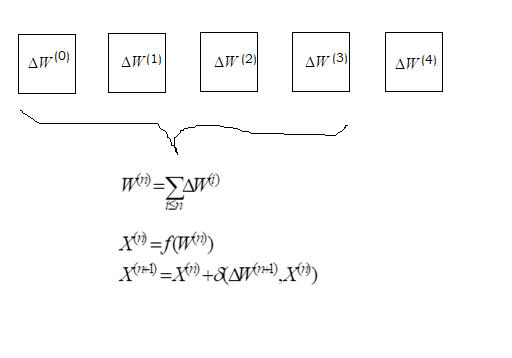
**Stochastic Stuff**

Memory stuff: formulas for Fn and pn. The generators: D, P. Discrete relations (just involves D): absolute and recursion relations for Fn and pn. Then continuum relations (just involves P): absolute and recursion relations for FL and pL. Smoulochowski equation – discrete and continuous.

Suppose we have a bunch of variables x(j) that are independently and identically distributed according to p(x) (like transfer matrix pieces).



We could write down the probability distribution of the set as Pn(Δ**W**) = p(ΔW(0))p(ΔW (1))p(ΔW­(2))…p(ΔW (n)) of course. But now let’s consider another related random variable X(n) = X(ΔW (0), ΔW (1), ΔW (2),…, ΔW (n)). We might be interested in the probability distribution function of *this* variable, or we might be interested in the average of a statistic, F(X(n)) of this variable:



But assuming direct integration is impractical, let’s consider the more general problem of determing pn(ξ) and/or Fn for general n, via a recursion relation. To solve the scaling equation, observe that we will need to know the initial value, i.e. p0(ξ), or F0 and this we ought to be able to calculate since these would be p(X(0)), and <F(X(0))> respectively.



**Scaling for an arbitrary function**

First we’ll consider the recursion relation for the X(n) variable. Suppose X(n) = X(ΔW(0), ΔW (1), ΔW (2), …, ΔW(n)) that follows the following recursion relation:



and a statistic we desire to calculate F(X(n)). Then:



where the dot stands for contraction of the two vectors. Defining the raising operator in this fashion



we can write:



Extending this to higher order, we should think of D(ΔW(n+1)­,X(n)­) as a raising operator. Or maybe just think of it like this: we can apply lower the X superscript in a function G, by compensating with raising the ΔW superscript in D. To apply it to an expression, you just lower all its X’s down one level to, say, m-1, and then tack on a D(ΔW(m),X(m-1)). As such,



Note that X always has the same superscript, no matter how far down we go. And we can’t go further than this. And besides X(0) is a function of ΔW (0) alone. Now taking the average, we have:



Now we’ll make some definitions below. Playing somewhat loose with the notation, D(X) coefficients are the moments of δ(ΔW,X) (averaged w/r to ΔW), where μℓ = <δ(ΔW,X)ℓ>ΔW. The notation obscures the fact that in reality, the second moment for instance would be a contraction <δ(ΔW,X)iδ(ΔW,X)j>∂2/∂X(i)∂X(j) if the ΔW(k) were matrices. But the notation is handy.



There is an alternate way to write this as well. We can then define an infinitesimal translation operator P:



Let’s look at P(X) a little more. We can write it as:



where the dot represents a contraction over the vector of values Xk and their increments δk(ΔW,X). And this would work out to:



Working out the first three terms…



If the <δ(ΔW,X)> were independent of X, then this would reduce to:



Its worth noting that the first two coefficients are the average and variance of the increment. Let’s take a look at the 2nd and 3rd coefficients…



So we can write this as:



There would be dots contracting the δ thing and X but I’ve left them off. So the coefficients of the derivatives in this case would just be the cumulants of δ(ΔW,X). One suspects this trend would continue, and from Wikipedia it does. Ln(<etX>) is the cumulant generating function. But note that though κ1,2,3 = moment about the mean, higher order κ’s do not. So it is not true that κn = <(x - <x>)n>. For instance, the Gaussian distribution’s cumulants are finite only up to second order. Again, neglecting the contraction notation for now, we could write this as:



where κn is the nth cumulant, and the { } stands for average of the anti-commutator. This means, you just work out the cumulant like it doesn’t depend on X, and then massage it back into X-dependent form. For instance, <δ3> would correspond to <δ3>∂3, while <δ2><δ> would correspond to [<δ2>∂2<δ>∂ + <δ>∂<δ2>∂2]/2 (basically just take average of permutations) and <δ>3 would correspond to <δ>∂<δ>∂<δ>∂. Well in any event, we can write Fn as:



So the important observation is that D(X(0))n, and exp[nP(X(0))] are ‘raising’ operators. Under the < >X(0) bracket, they raise the argument of any function F(X0) to F(Xn), again, under the < >X(0) bracket. Apropos, we can develop a few recursion relations…



Or,



From which we have:



and if we want to play loose perhaps with derivatives, then we could say:



I’m thinking that this expression is legitimate even though the series is technically only defined for integer n; because it seems we could smoothly analytically continue it to real n based off of the exponential operator expression above. This would technically require raising the D operator to a fractional power, which would mean fractional derivatives and such. These also are well defined though – in terms of integrals I believe. Maybe it’s OK. Note that we cannot take the < > off to make this a differential equation for Fn alone because it was only with the < > that we were able to collapse the D(ΔW1,X0)D(ΔW2,X0)…D(ΔWn,X0) into a single D. Now let’s consider some equations for the probability distribution. Let’s do the absolute equation first:



and so we’d have:



Where Dn† is the adjoint of Dn, i.e., it is the operator we get via repeated integration by parts, to put all the derivatives on p0(X).



where we have defined the adjoint operator D†(X) to be the operator you get when you apply repeated IBP to get the derivatives off the δ and onto the pn(X). So,



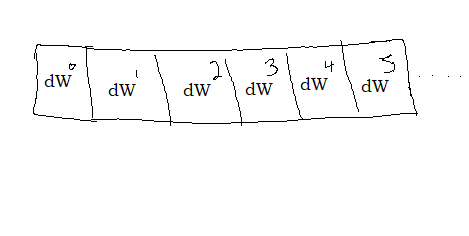
It would seem that p(ξ) is a special function in that its recursive relation can be written in terms of itself, and not the average of the product of itself and some auxiliary function. We could do a z-transform to get rid of the difference-equation awkwardness, solve the ODE, and then invert the transform, but whatever. And the questionable derivative equation would be:



where P† is the ‘adjoint’ of P, analogously defined. But observe that D, and D† both have infinite number of terms, unless for some reason it terminates because μj or κj terminate (the latter does for the Gaussian distribution) or because F, p are polynomial in form and so naturally get annihilated after a certain number of derivatives.

**Going to the continuum**

Problem so far is that the derivatives don’t stop, unless higher moments μk, or κk vanish at some point for some reason, and so we cannot completely define D or P, and so cannot get a closed recursion relation. But a simplification is afforded us if we go to the continuum limit.



Particularly as applied to time-dependent, (or length-dependent) processes, we may cut up the interval 0 to L into n = L/δL infinitesimal step sizes, δℓ, which covers a random variable dW, and apply the formalism above to those pieces, eventually taking the limit δL → 0, keeping L fixed. Now let’s go back to:



And this is intuitive in the context of the transfer matrices. Then:



where the dot stands for contraction of the two vectors. So far this ought to make sense as well. Then we’ve got the usual:



(where X(0) is just the first entity, but still has ‘width’ δL). Then take the average and we have, subtracting δL from the exponent, to make it look nicer:



And then we can introduce P to get:



And this is where it gets perplexing. Now we would expand P(X(0)) in powers of δL,



and take the small δL limit, keeping L fixed. And note that X is really just a dummy variable in this processes. I suppose that the way we evaluate it is to (at least in the transfer matrix context) let dW(δL) be a sort of macroscopic element ΔW(ℓ) or something. And then we take the small ℓ limit of the expectation, which I presume would mean that we keep just the O(n) terms, and make some connection ℓ ~ n. But unit wise, oughtn’t we rather make the correspondance n ~ ℓ/δL. And if we did this, then even in the small ℓ limit, P wouldn’t converge to anything proportional to δL, rather it would blow up. The averages must also be rescaled I guess.



Well, suppose we consider N numbers, uniformly ranging between 0 and 1. For the average we’d say:



And if we did a continuum approximation:



And we’d see that we must have <δy> = δy/2. So the



Well, in any event, then we’d have:



and upon differentiation w/r to L,



Or in other words:



So this is the analogue to our end result in the Stochastic file.



So that:



Recall λ determines where along the interval (t,t+dt), the functions are evaluated. And a, b are determined vis a vis dX/dt = a + bw. The value of λ depends on how this equation is defined. I’d say λ = 0, if we want to compare with Mello.

Again, X(L) is just a dummy variable, in the sense that PδL(X(0)) and PδL(X(L)) have the same functional form, obviously. Note that we could continue this pattern and write down a Taylor series expansion of FL. Differentiating the highlighted equation α times gives us:



and so we can say:



Finally, plugging FL(X – ξ) = δ(X – ξ) into the first equation gives us:



and so we get:



the original differential equation gives us:



So we have:



But observe that there is nothing as yet which restricts the number of terms in the PδL† expansion: it could contain an infinitude of derivatives. Still, in practice I think one finds that the derivatives will terminate past 2nd order; certainly white noise does. But, a δL expansion on the averages means that we still need to consider M to basically all orders to extract the Nth order term in any of the expectations. That’s why we do the M­NMN† expansion. And we found expressions for the averages out to order N, under certain approximations on how the individual matrices average.

**Integral equation approach**

Another approach is to develop an integro-recursion relation. This has the advantage of there not being an infinite number of terms. Now we’re specializing to the present recurence relation:



Then for a function F(X) we would have:



Let’s consider F(X(n+1)) = δ(X(n+1) – ξ). Then we’ll have:



And now, we got:



and the bare p(ΔW) is the probability distribution of the building block. This is called the Smoulochowski equation. Now suppose we wanted to get a differential equation out of this expression – the same one we’ve already got of course. We can expand in powers of δ,



which we can write as:



Be sure to note that the perturbative expansion has no reason to stop as yet. A case would have to be made. Now let’s go to the continuum. Our recurrence relation would translate to:



Then,



And then,



Perhaps to be clear, we should write out the length to which these probability distributions refer. Then we’d say:



and pδL(ΔW) is the probability distribution of the building block. Now suppose we wanted to get a differential equation out of this expression – the same one we already got of course. Then we’ll have:



which we can write as:



And then typically we can expand these averages in powers of δℓ. So again, in order to get a finite order PDE, we may have to technically assume a weak scattering limit, or something, to ensure that the numerators get progressively higher in orders of δℓ. Note that if we have a matrix model of M, going out to first order in δL would still probably require the entire matrix (to some degree), as we’ll recall that contributions to order δL, mean contributions to order N. And in fact we found that going out to that order would renormalize the ‘average’, i.e., <M11,ab(1)M11,ab(1)†> had different index dependence than O{<M11,ab(N)M11,αβ(N)†>,N} did (where O{x,N} means x to order N).

**Special approaches pertaining to product relation X(n) = ΔW(0)ΔW(1)…. ΔW(n)**

Another approach, feasible for many types of recursions, is to develop an integro-recursion relation. This has the advantage of there not being an infinite number of terms. Now we’re specializing to the present recurence relation:



Then for a function F(X) we would have:



Let’s consider F(X(n+1)) = δ(X(n+1) – ξ). Then we’ll have:



where the division means we’re multiplying by the inverse on the right. Now the measure dμ(M) for the *transfer* matrices is invariant so we can write:



which we might write as, being clear about the order (n) to which the probability distributions refer…



and the bare p(ΔW) is the probability distribution of the building block. This is called the Smoulochowski equation. This should make quite a bit of sense as for to get matrix ξ out of the two pieces x and x´, we must have that x´ = ξx-1. And the probability of getting it is simply the sum of all ways to do it, i.e., sum over all ξΔW-1 probabilities, which is p(ΔW)p(ξΔW-1), integrated over all ΔW’s, which is effectuated by measure dμ(ΔW). We could turn this into a pure integral equation instead of a difference-integral equation by using the z-transform. Applying the z-transform to both sides…



and so,



Now suppose we wanted to get a differential equation out of this expression – the same one we’ve already got of course. Well, we can go backwards and write ξ/ΔW = ξ + Δ(ΔW,ξ) (this isn’t necessarily δ(ΔW,ξ)). Then we’ll have:



which we can write as:



We could convert this to a differential equation by taking the z-transform. We would get the same result if we perturbatively expanded the integral equation of p(z,ξ) – it’s just an interchange of operations. So anyway the result would be:



Be sure to note that the perturbative expansion has no reason to stop as yet. A case would have to be made. Now let’s go to the continuum. Our recurrence relation would translate to:



Then,



where the division means we’re multiplying by the inverse on the right. Now the measure dμ(M) for the transfer matrices is invariant so we can write:



Perhaps to be clear, we should write out the length to which these probability distributions refer. Then we’d say:



and p(ΔW) is the probability distribution of the building block. Now suppose we wanted to get a differential equation out of this expression – the same one we already got of course. Well, we can go backwards and write ξ/ΔW = ξ + Δ(ΔW,ξ) (this isn’t necessarily δ(ΔW,ξ)). Then we’ll have:



which we can write as:



And then typically we can expand these averages in powers of δℓ. So again, in order to get a finite order PDE, we may have to technically assume a weak scattering limit, or something, to ensure that the numerators get progressively higher in orders of δℓ. Note that if we have a matrix model of M, going out to first order in δL would still probably require the entire matrix (to some degree), as we’ll recall that contributions to order δL, mean contributions to order N. And in fact we found that going out to that order would renormalize the ‘average’, i.e., <M11,ab(1)M11,ab(1)†> had different index dependence than O{<M11,ab(N)M11,αβ(N)†>,N} did (where O{x,N} means x to order N).

**Constraints among variables**

Suppose there are relations among the variables ΔW(n)i so that they are not all independent. This is the case for the elements of the transfer matrix M for instance. Consider for example the 1D 2×2 transfer matrix,



where M12 = M21\*, and M11 = M22\*. So that limits us to four, instead of 8, d.o.f. But there are actually just three due to the flux conservation constraint, which the matrix elements do not *obviously* satisfy. And the increments are:



So we’ll note there are two d.o.f. here, but the general Nth order matrix will have three, as the 2N d.o.f. explicit under repeated multiplication, will combine in such a way as to form only three. What if we put it in terms of the Δ+- and Δ++? The recursion relation is:



The translation generator is:



So we have:



Even if we stop here (perhaps acceptable in the weak scattering limit whereby we stop after second order in V?), it looks like we have 16 freaking terms to consider. So the first 8 terms are:



and the second 8 terms are:



So altogether we have:



which can be simplified:



and I don’t really want to figure this out. Perhaps later if there appears to be a pattern. And so then to this order, our scaling equation would read:



Now suppose that we set ξ22 = ξ11\*, and ξ21 = ξ12\*, then:



And we’ll observe that now pn(ξ) is a function of the ξ’s and their complex conjugates – as seems to be necessary in general. And the PDE operator is real and so presumably preserves the reality of p(ξ11, ξ11\*, ξ12, ξ12\*). So its like this. Consider a system with two variables x, and y. But suppose y = x2. So there is only one d.o.f. and so probability distribution function is a p(x). But the set up above, doesn’t incorportate this symmetry directly into the problem. So it gives us a way to calculate p(x,y) such that when set y = x2, we get the desired p(x). It would be nice to be able to convert the PDE in terms of x, y to an ODE just in terms of x, for instance.

How could we turn this into a PDE w/r to u, υ, and λ? Let’s consider an example p(x,y), where y = f(x). If we have a PDE for p(x,y), what would be the PDE for p(x)? Well suppose:



Well we can say:



Not enough equations to put the partials in terms of x derivatives yet. What if I calculate:



which constitutes an extra equation? Don’t know b/c I could generate as many such equations as I want.

**Scaling of X(n+1) = X(n) + ΔW(n+1)**

Consider some independent c-number random variables ΔW(j). Let these be independently distributed according to some probability distribution p(ΔW). And consider a related random variable X(n), defined for now like this:



Now let us consider a statistic F(X). F(X) might be X itself for instance, or maybe X2, or 1/(1+X), whatever. We’d like to devise a means to calculate <F(X)>, where average means average over all ΔW(j) according to their probability distributions. For instance if F(X) = X, then we’d just do:



Or if F(X) = X2, then we’d have:



These formulas give explicitly the dependence of <F> with n. Now let’s see if we can develop a difference/differential equation for <F>. In general, regardless of the functional form of F we have:



if F(X) = X, then only the first derivative would be non-vanishing, and we’d have:



Taking the average of both sides we’d get:



and we could certainly solve this recursion relation for <X(n)> in general. And we could do the same for F(X) = X2. In the process we would have to know initial conditions <X(0)>. But since X(0) = ΔW(0), this would be known. So this is one way. But suppose F(X) were some arbitrary function whose derivatives didn’t vanish? Can I put this in his notation?



The first statement would read:



and the second,



Now the first relation takes a function of X(1) and converts it to a function of X(0). Note that in the second line we have a function of X(1). We can apply the first to the second, and get:



What does this look like exactly?



which is certainly true. Suppose I were to take the average at this point. Then what?



where X(0) is typically ΔW(0) itself. Since ΔW(1) and ΔW(2) are independent, and not part of X(0), we can say that:



and we could write this in short hand as:



Extending this to higher order:



And then also,



We can then define a translation operator P so that:



at which point we could write:



We could turn this into a differential equation so as to get:



and using the definition of the translation operator we could write this in turn as:



Applied to our sum example, what does this say? Well what is the translation operator?



Now in our case, F(X) = X, only up to 1 derivative will apply to it. So we can write this as:



which itself need only be expanded to linear order, to get:



So then we would have:



The derivative gives 1, and upon integration we just get:



Evaluating the constant of integration via <F(X(0))> = <ΔW> gives us our answer:



which is correct. What about the variance? In that case we can keep up to two derivatives in the ln, and expand out to second order:



Applying this to F(X) = X2 we have:



Here we can use our previous results on <X(n)> to fill in…



Integrating we get:



Using <F(X(0))> = <ΔW2> we have:



Filling this in and simplifying we come to:



and this is also correct! What if F(X) is an arbitrary function? Then we have:



Applied to our sum example, what does this say? Well what is the translation operator?



where κj is called the jth cumulant of ‘whatever’. So applying this we have:



Now a simplification occurs in two cases. First, if the ΔW(j)’s are Gaussian distributed with a zero mean, then only the second cumulant survives – all higher cumulants will be 0. And so then we’ll get:



Even if the ΔW’s are not so distributed, a simplification will occur if they are weak. He says consider a sample of fixed length L, with n scatterers and spacing δx between them. Let δx → 0, so that n → ∞ so that L stays constant. And let κ2/δx remain constant at σ2 I suppose, which would imply that κ2 itself is going to 0 (this is the weak part). In that case, going back to our problem we have:



Now suppose that the rest of the cumulants look like this κr ~ κ2r/2. Then in that case

κr/δx = (κ2/δx)κ2(r-2)/2. And so then if r > 2, all of these cumulants will go to zero identically in the continuum limit. And so then all we’ll be left with is:



Suppose we’re looking at the probability distribution of these random variables. Then we’d have

F(X) = δ(X – ξ). Filling this into the expression above we have:



where in the last we assume boundary terms are negligible. And then evaluating the integral we would have:



So we can write the evolution equation for the probability distribution in closed form! Do we get a Gaussian out of this though? We can do a Laplace transform on the L variable to get:



Adding them all up we get:



And this is indeed the solution. So the variance grows with length, as it should since we have more random variables. Now let the ΔW’s, and X’s be random vector variables. Still we have:



Applied to our sum example, what does this say? Well what is the translation operator? First, I’m going to make it implicit that when doing a Taylor series expansion of a vector function it is:



So then,



and in general we can write this as:



Now suppose we take the continuum limit. Let δx be the spacing between scatterers ΔW(n)μ and ΔW(n+1)ν. And let this go to zero, as n goes to infinity. Further, let the strength of the scatter κμν go to zero such that κμν/δx = σμν.



Now suppose that the rest of the cumulants go to zero faster than κμν, then this will be the only one left, and we’ll have:



**Comparing integral and continuum approach**

Let’s see if we can get the same thing using the integral equation approach, and the differential equation approach. So we have:



And I’ll take:



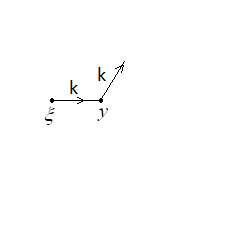
Let’s start with the integral equation:



Taking the z-transform,



where p0() is the log-normal distribution. Drawing out that Fourier transform diagram thing…



and taking the Fourier transform of both sides we have:



And the Fourier transform of the initial distribution is:



So then we take the inverse Fourier transform, and then inverse z-transform. Maybe it would be best to reverse the order in which this is done. Then,



So then we have:



Taking the inverse transform, we get:



Which is right. Now let’s consider from the differential perspective. We have:



Or



Now,



Now to evaluate the cumulants. The approach, as I guess it, is to do the cumulants in general, but then just extract the O(n) term, and then basically replace N with δL. So,



And all higher order cumulants vanish. So,



And our equation becomes, forming the conjugate of P,



Taking the Laplace transform w/r to time, and Fourier transform w/r to position:



Taking the inverse Laplace transform, gives:



And then the inverse Fourrier transform,



Which is correct. But what if the series weren’t Gaussian distributed?

**Scaling of series X(n+1) = X(n)ΔW(n+1) with uniform distribution in [0,1]**

Consider the following problem. Let a and b be uniformly distributed between 0 and 1. What is the probability distribution of their product? Well..



Well that’s interesting. And is this normalized,



So p(ξ) = -ln(ξ). Huh. That means it’s highest at ξ = 0. I guess that makes sense, because the product of any two numbers between 0 and 1 is going to be smaller than either. And so the distribution will be biased towards 0. Suppose I continue multiplying numbers together. What about a third term?



One might expect that



Checking on Mathcad, this distribution is also integrable and normalized. Can I get the probability distribution of the product of n variables ΔW(n)? What if we look at the integral equation itself? Have to be carefull here because the measure is not invariant w/r to scaling so going back to basics:



As ξ → 1, we can infer that pn+1(ξ) → 0. Let’s take the z-transform of both sides:



Now we have just a normal integral equation. Let’s differentiate both sides:



And this equation can be solved. The answer is:



What is the arbitrary constant? Well we know that p(n,ξ = 1) = 0. So perhaps p(z,ξ = 1) = 0. Yes I’d say so. Well in that case we’d just get C = 0. Well I’ll note that p0(1) doesn’t equal 0. So let’s look for another condition. Perhaps we can use a normalization condition. Consider requirement that p be normalized, how does this relate to its z-transform?



Applying this then,



Great! And so then the solution to our integral equation is:



Isn’t the integrand analytic, except at z = 0? I think so. Then we only need the residue at z = 0. So we just need to expand this integrand in powers about z = 0. So,



And the residue is therefore:



and so we have:



For instance, the formula for n = 2 would be…I’ve got to review how to do these complex integrals. What changes if the limits of integration change? And if p(ΔW) changes? Let p(ΔW) be non-uniform but keep limits of integration:



Now apply Z-transform,



If p0 is any sort of power distribution, then it would seem that this can be done.

**Differential equation approach in X variable**

*Now let’s examine converting this to a differential equation.* How might this proceed? So consider the identically distributed random c-number variables X(n), following the definition:



The recursion relation is:



and P(X(n)) is given by:



Well the first couple moments are…



Let’s just keep the first two. Then,



and so the evolution equation of the probability distribution is:



Try separation of variables:



Now try power solutions:



And so solution is:



The initial condition it must reduce to is:



This could just mean that we pick out the k that makes the power = 0. And set the other term to 0. For instance,



And then choose A+ = A, and A- = 0. And so then we can write:



Now constant A has to be chosen so that p is normalized, but this would make p(n,ξ) = 1. And this basically means that the distribution remains uniform? Harumph! This can’t be correct b/c p(n,ξ) = 1 doesn’t satisfy the PDE. What if I take a LT of both sides of the PDE?



Now let’s try a power solution: p(s,ξ) = ξp for the homogeneous part:



And the homogeneous solution is: p(s,ξ) = C. What is this?



So general solution is:



The boundary conditions are what? I don’t know. We have to examine the integral equation to figure this out. We know that p(n,ξ) → 0. The Laplace transform should also satisfy this property. So we know that p(s,ξ = 1) = 0. So this would imply,



Filling this in we get:



We also have the normalization condition. Now we know that:



So this is what the normalization condition works out to be for p(s,ξ). Let’s apply it:



Filling this into p(s,ξ) we have:



Now we have:



Making these simplifications we can write:



And so then,



Whatever. Well we have the other solution anyway. I guess this just means that keeping just those two derivatives is insufficient to adequately describe the solution.

**Differential equation approach in lnX variable**

On the other hand, he suggests that we may write the recursion relation as:



And then translation operator is:



Well the first couple moments are…



And the moments keep growing. This interestingly shows that we cannot just drop terms based off of numerical coefficients because one can change variables and make them go the other way (increase instead of decrease). You would have to have some way of estimating the derivative as well. Well if we were to keep the first two terms anyway…



And so the scaling equation would read:



If we do separation of variables again,



So the general solution is:



This is the same thing, naturally I suppose.

**Continuum differential approach in X variable**

Since the solution has a factoral,



I’d like to see if a continuum approach can even be made for this scenario. If not, then I don’t see how it could transpire that the continuum approach could work for the transfer matrix approach either. Conversely, since it does (?), I’d like to see how it happens. Starting with:



Now to evaluate the cumulants. The approach, as I guess it, is to do the cumulants in general, but then just extract the O(n) term, and then basically replace N with δL. So,



And then the first cumulant is:



to evaluate it we evaluate the expectation of a product of ΔW(n)’s and take the small n limit, replacing n with δL in the process. So then,



(assuming I can make this complex do this with a complex number)…So then I’d say,



But we already have a problem since this isn’t proportional to δL. And so PδL would blow up! Next guy is, for what it’s worth…



Working on the expectation…



So we’d have:



But still, the first term is problematic. Consider starting from the integral equation:



Now let’s make an expansion about ΔW = 1.



And so if we were allowed to stop here, we’d have:



This will also be problematic. The first coefficient, which should give me 1, gives:



Perhaps I should check the next term? But this will seem to oscillate back and forth.

**Scaling of series X(n+1) = X(n)ΔW(n+1) with uniform distribution in [0,2]**

Consider the following problem. Let a and b be uniformly distributed between 0 and 2. What is the probability distribution of their product?



Let’s take the z-transform of both sides. But we have a problem because we have n in the limits of integration, and so the Z transform of the RHS is problematic. Could take a derivative w/r to ξ, but then I’d have functions with different arguments. Not sure if this is legal, but perhaps I can go back to finite interval case. Let,



And let’s consider the integral equation:



Then we could take the Mellin transform of both sides to get:



It would seem that we can now solve the difference equation:



The Mellin transform of this function is:



This would be defined for all s in the R.H.P. And then we’d take the inverse transform,



It would seem that if the exponent is negative, that I would just get 0, because I could close the integral to the right, and I wouldn’t be enclosing any poles. So perhaps the exponent must be positive so that I’m forced to close to the left, and enclose the pole at s = 0. The exponent would be positive if:



This is good news because we know from above, and on general grounds, that ξ is bounded by this. Proceding then,



So we have:



Is this correct? Well, p0(ξ) = ½. And p1(ξ) = (1/4)ln(4/ξ). Let’s check,



All these match. So this appears to be correct. But this wouldn’t exactly seem to have a straightforward continuum analog. Let’s do this another way, by working with the ln of the series. If I change to Y = lnX. Then,



Then starting with the integral equation:



and taking the Fourier transform of both sides we have:



And then the inverse transform would yield:



But what is the initial transform? Well if p0(x) = (1/2)θ(0<x<2), then p0(y) = (1/2)θ(0<ey<2)ey = (1/2)θ(-∞<y<ln2)ey. And its Fourier transform would be



So then,



Now the pole is at k = -i, and so for y > ln2n+1 we would close up and get zero. For y < ln2n+1 we would close down and get that residue,



And so for the y-variables we get:



Going back to the x-variable we have:



Which matches. Yay! But now consider: the sum of independent variables ought to be made into a Gaussian. So can this be done here? Well



The extremum is at:



The value of the exponent at the extremum is:



And the second derivative at this extremum is:



And so we can say:



So indeed, the Gaussian form is a good approximation. And what is the average and variance of y?



This is just about exactly what we should expect. Just missing an extra factor of ln2 in the average, but the average is correct up to O(n). Perhaps that’s all we should expect? So a straightforward continuumization seems possible. Translating back to X then, we have:



Note that the Gaussian approximation here is good near the local max, but obviously fails near the boundaries – tails. Note we can get pn(x) directly too, from the fact that:



where Y is a normally distributed variable. And so,



Let’s consider sum of independent distributed variables -∞ < y < 0, via P(y) = eydy. It appears the distribution of a sum of n such variables is:



So from experience we know that this may be approximated by a normal distribution for large n. But in what sense is this accurate? What if we were to consider the standard Z = (1/n)Σy - <y>.

**Continuum differential approach in x variable**

And now I’d like to check this against the continuum limit. The hope is that now it might work since x = 1 is the average, and so there would now be a tendency for the next term in the series to be close to the former one, unlike in the previous sequences where the average was not unity.



Now to evaluate the cumulants. The approach, as I guess it, is to do the cumulants in general, but then just extract the O(n) term, and then basically replace N with δL. So,



And then the first cumulant is:



to evaluate it, we evaluate the expectation of a product of x­(n)’s and take the small n limit, replacing n with δL in the process. So then,



So then I’d say,



Okay...next guy:



Working on the expectation…



This also doesn’t vanish in the small n limit. What if we go back to the expansion stemming from the integral equation? We’d have:



And the <(x-1)2> terms do not vanish, and apparently have O(1) parts anyway. So it doesn’t seem to work.

**Continuum differential approach in lnx variable**

We would expect this approach to work, or that it would at least give the meat of the distribution, as has been demonstrated before when analyzing the Y = y0 + y1 + … + yn distribution. But again, we might expect failure at the tails.

**Scaling of series X(n+1) = X(n)ΔW(n+1) with exponential distribution in [0,∞]**

What if we switched to a probability distribution that was infinite, or semi-infinite but still had an average of one?



Then we could take the Mellin transform of both sides to get:



It would seem that we can now solve the difference equation:



And then I should take the inverse Mellin transform. So,



To be more explicit, I guess I need to assume a probability distribution for p0(x). So let’s use some



Which is normalizable and has the requisite average.



So p0(x) = e-x. The Mellin transform of this function is:



Note that I cannot assume integer s. And then we’d take the inverse tansform,



I presume that Γ(s)n+1 decays for –s more quickly than ξ-s grows. The analytically continued Γ(s) is:



and so it’s residues at the poles along the negative real axis are:



Therefore what are

But then we have an awkward integral to perform. Cannot close to right since Γ(s) grows to fast in that direction. But cannot close to right because e-slnξ grows in that direction, unless Γ(s) damps it? Check out the analytic continuation of Γ(z) to l.h.p., we might be able to close that way, and use its poles at the integer z’s. Let’s go back to the exponential p0(x) this time, but using the Fourier transform method. Then we have…



But what is the initial transform? Well if p0(x) = e-x, then p0(y) = e-exp(y)ey. And its Fourier transform would be



And so then,



So I get the same thing. Don’t think this is particularly easier than working with the Gamma function version of this problem.

**Will integral & continuum approaches match up for non-Gaussian sum series**

Let’s reprise,



where



Then starting with the integral equation:



and taking the Fourier transform of both sides we have:



And then the inverse transform would yield:



And eventually,



Then far enough away from the tails we could say:



Which compares very well to what we would expect on central limit theorem grounds since:



Now the question is, could I arrive at the same central-limit result by doing a continuum approximation? Well we have:



Now to evaluate the cumulants. The approach, as I guess it, is to do the cumulants in general, but then just extract the O(n) term, and then basically replace N with δL. So,



And how does the cumulant of a sum of independent variables behave? Let’s consider:



So this seems to indicate that I’ll just get:



But the third cumulant will not vanish for this distribution. And so it seems that there would be another term added to the P operator. And yet, this wouldn’t make sense since the distribution must converge to a Gaussian, which was ensured by the operator going out to order 2. So am I taking the continuum limit incorrectly?



And our equation becomes, forming the conjugate of P,



Taking the Laplace transform w/r to time, and Fourier transform w/r to position:



Taking the inverse Laplace transform, gives:



And then the inverse Fourrier transform,



**Comparing OU process with Mello’s formulation**

So we have for an OU process, say Ito,



Now according to conventional Taylor series expansion, we’d have:



Taking average and all,



and this is true. But then what about Mello’s other prescription, where we form the P-operator,



and,

